Chapter Three
Three Partial Solutions to Hilbert's Seventh Problem.

We recall from Hilbert's address that he considered it to be a very difficult problem to prove that

the expression α^β*, for an algebraic base and an irrational algebraic exponent, e. g., the number* $2^{\sqrt{2}}$ *or* e^π*, always represents a transcendental or at least an irrational number.*

The above quotation is one of the most important comments Hilbert made while posing his seventh problem. We earlier pointed out the Hilbert's seventh problem is usually thought of as being the transcendence of α^β when $\alpha \neq 0, 1$ and the irrational number β are both algebraic. And the numbers that are most often used to illustrate this result are those given above by Hilbert; the numbers $2^{\sqrt{2}}$ and e^π. Yet each of these numbers have additional properties that were exploited to demonstrate their transcendence several years before the solution to the more general problem. In this chapter we discuss in some detail the Russian mathematician A. O. Gelfond's proof of the transcendence of e^π, and then briefly consider another Russian mathematician's, R. O. Kuzmin, proof of the transcendence of $2^{\sqrt{2}}$ and the German Karl Boehle's generalization of these two results.

The first partial solution to Hilbert's seventh problem

Theorem (Gelfond, 1929) e^π *is transcendental.*

Before we look at Gelfond's proof, which is the most technically challenging proof we will consider in these notes, we note that, as Gelfond pointed out at the end of his paper, his method could be modified to establish the following more general result:

Theorem. *If* $\alpha \neq 0, 1$ *is algebraic and* r *is a positive rational number then*

$$\alpha^{\sqrt{-r}} \text{ is transcendental.}$$

This more general result implies the transcendence of both $2^{\sqrt{-2}}$ and $e^\pi = i^{-2i}$ but not of the other number Hilbert mentioned, $2^{\sqrt{2}}$.

Gelfond needed a new idea–the Hermite and Lindemann idea of using a suitable modification of the power series of e^z fails when used to study e^π. And it is easy to see why–any truncation of the power series representation for e^π will be a polynomial in π. According to Lindemann's Theorem π is transcendental so Gelfond could not apply the idea of using this power series to obtain an algebraic number, and so eventually an integer strictly between 0 and 1.

Outline of this failed proof. Suppose e^π is algebraic and satisfies an integral, polynomial equation

$$r_0 + r_1 e^\pi + r_2 e^{2\pi} + \cdots + r_d e^{d\pi} = 0, r_d \neq 0. \tag{1}$$

For positive integers $N' > N$ we separate the power series for $e^{\pi z}$ into three pieces:

$$e^{z\pi} = \sum_{k=0}^{\infty} \frac{(z\pi)^k}{k!} = \underbrace{\sum_{k=0}^{N} \frac{(z\pi)^k}{k!}}_{\text{Main Term, } M_N(z\pi)} + \underbrace{\sum_{k=N+1}^{N'} \frac{(z\pi)^k}{k!}}_{\text{Intermediate Term, } I_{N,N'}(z\pi)} + \underbrace{\sum_{k=N'+1}^{\infty} \frac{(z\pi)^k}{k!}}_{\text{Tail, } T_{N'}(z\pi)}$$

We use this expression at the values $t = 1, \ldots, d$ and the presumed polynomial relationship which vanishes at e^π, (1) above. Even if we arrange things so that these Intermediate sums all vanish, upon replacing $e^{t\pi}$ by the sum $M_N(t\pi) + T_{N'}(t\pi)$ in the relationship above we would obtain:

$$r_0 M_N(0) + r_1\big(M_N(\pi) + T_{N'}(\pi)\big) + r_2\big(M_N(2\pi) + T_{N'}(2\pi)\big) + \\ \cdots + r_d\big(M_N(d\pi) + T_{N'}(d\pi)\big) = 0,$$

Which leads to the equation:

$$r_0 M_N(0) + r_1\big(M_N(\pi)\big) + r_2\big(M_N(2\pi)\big) + \cdots + r_d\big(M_N(d\pi)\big) \\ = -\Big(r_1\big(T_{N'}(\pi)\big) + r_2\big(T_{N'}(2\pi)\big) + \cdots + r_d\big(T_{N'}(d\pi)\big)\Big).$$

This is an equality of two transcendental numbers, because Lindemann had already demonstrated the transcendence of π. Thus the classical approach of using an equation as above to produce a nonzero, rational integer less than 1 cannot work.

Gelfond's proof of the transcendence of e^π.

At the heart of Gelfond's proof is *not* the power series representation for e, or more accurately for the function $e^{\pi z}$, but other polynomial approximations to $e^{\pi z}$. These polynomial approximations are based on the Gaussian integers, so we briefly begin with them.

Key new ingredients in Gelfond's proof.

1. The collection of Gaussian integers is the set $\{a + bi : a, b \text{ integers}\}$. The crucial point about the Gaussian integers is that *if* e^π is assumed to be an algebraic number then the function $f(z) = e^{\pi z}$ will take on an algebraic value at each of the Gaussian integers. Specifically, if $a + bi$ is a Gaussian integer, then

$$f(a + bi) = e^{\pi(a+bi)} = e^{\pi a} \times e^{i\pi b} = (-1)^b (e^\pi)^a$$

is algebraic. We will discuss the more subtle properties of the Gaussian integers that Gelfond exploited as we present his proof of the transcendence of e^π, but in order to even describe *how* Gelfond used them to give the polynomial approximations to the function $e^{\pi z}$ we need to begin with a way to order them. Gelfond ordered the Gaussian integers by their modulii, and for Gaussian integers with equal modulli by their arguments. This yields the following ordering:

$$z_0 = 0, z_1 = 1, z_2 = i, z_3 = -1, z_4 = -i, z_5 = 1 + i, \\ z_6 = -1 + i, z_7 = -1 - i, z_8 = 1 - i, \ldots$$

2. It is possible to approximate the function e^z by an infinite series each term of which is a polynomial all of whose zeros are among the (ordered) Gaussian integers (this leads to the so-called *Newton series* of the function $f(z) = e^{\pi z}$). Let $\{z_0 = 0, z_1, z_2, \ldots\}$ denote the (ordered) Gaussian integers and consider the polynomials

$$P_0(z) = 1, \quad P_1(z) = z = z - z_0, \quad P_2(z) = z(z - z_1), \ldots, \\ P_k(z) = z(z - z_1) \cdots (z - z_{k-1}).$$

Then we have an *interpolation* to the function $f(z) = e^{\pi z}$ by these polynomials:

$$e^{\pi z} = A_0 P_0(z) + A_1 P_1(z) + A_2 P_2(z) + \ldots + A_n P_n(z) + R_n(z), \tag{2}$$

where the numerical coefficients $A_0, A_1, \ldots A_n$ and the polynomial remainder term $R_n(z)$ have integral representations. In particular, if γ_n and γ_n' are any simple, closed curves that enclose the interpolation points $z_0, z_1, \ldots, z_n$ then

$$A_n = \frac{1}{2\pi i} \int_{\gamma_n} \frac{e^{\pi \zeta}}{\zeta(\zeta - z_1) \ldots (\zeta - z_n)} d\zeta$$

and

$$R_n(z) = \frac{P_{n+1}(z)}{2\pi i} \int_{\gamma_n'} \frac{e^{\pi \zeta}}{\zeta(\zeta - z_1) \ldots (\zeta - z_n)} d\zeta.$$

We will come back to the important point of choosing these contours in the proof below.

It is perhaps worthwhile to look at the above Newton interpolation to the function $e^{\pi z}$ in terms of the ideas of the last chapter.

$$t = z_0 \quad e^{\pi t} = \underbrace{A_0P_0(t)}_{\text{Main Term}} + \underbrace{A_1P_1(t) + \cdots A_NP_N(t)}_{\text{Vanishing Intermediate Term}} + \underbrace{R_N(z)}_{\text{Tail}}$$

$$t = z_1 \quad e^{\pi t} = \underbrace{A_0P_0(t) + A_1P_1(t)}_{\text{Main Term}} + \underbrace{A_2P_2(t) + \cdots A_NP_N(t)}_{\text{Vanishing Intermediate Term}} + \underbrace{R_N(z)}_{\text{Tail}}$$

$$\vdots \qquad \vdots$$

$$t = z_k \quad e^{\pi t} = \underbrace{A_0P_0(t) + \cdots A_kP_k(t)}_{\text{Main Term}} + \underbrace{A_{k+1}P_{k+1}(t) + \cdots A_NP_N(t)}_{\text{Vanishing Intermediate Term}} + \underbrace{R_N(z)}_{\text{Tail}}$$

$$\vdots \qquad \vdots$$

So if $k' > k$ then the polynomial approximation to $e^{\pi z_{k'}}$ involves a polynomial of higher degree than the polynomial approximation to $e^{\pi z_k}$.

Now that we have this representation of the function $e^{\pi z}$ we can outline Gelfond's proof (we will expand upon and justify each step below):

Outline of Gelfond's Proof.

Step 1. Assume e^{π} is algebraic (so the function $e^{\pi z}$ assumes an algebraic value at each Gaussian integer).

Step 2. Show that for n sufficiently large $A_n = 0$. This means that there exists a positive integer N^* so that if $n > N^*$, $A_n = 0$. This tells us that for all $n > N^*$ we have the representation for the function $e^{\pi z}$:

$$e^{\pi z} = A_0P_0(z) + A_1P_1(z) + \cdots + A_{N^*}P_{N^*}(z) + R_n(z).$$

Step 3. It follows upon letting γ_n' be a circle of radius n and letting $n \to \infty$ that $R_n(z) \to 0$ for all z. Therefore the function $e^{\pi z}$ may be represented by a polynomial.

Step 4. Conclude that the function e^z is **not** a transcendental function.

This last conclusion contradicts the transcendence of the function e^z and so shows that our initial assumption, that e^{π} is algebraic, cannot hold. Thus e^{π} is transcendental.

Details of Gelfond's Proof.

Clearly Step 2 is at the heart of Gelfond's proof, so we first focus first on this. His demonstration that $A_n = 0$ for all n sufficiently large is ingenious and has two parts. The first part uses analytic tools and it provides an upper bound for $|A_n|$ which depends, in part, on choosing a reasonably short contour of integration γ_n. The second part is entirely algebraic in nature and involves taking the norm of an algebraic number. This is the only part of the proof that uses the assumption that e^π is algebraic. Under this assumption each of the expressions A_n is an algebraic number. Using fairly subtle estimates Gelfond finds a small denominator for A_n. If $A_n \neq 0$ then multiplying A_n by a denominator and taking the algebraic norm contradicts the upper bound obtained in the first part. Thus $A_n = 0$.

Step 2. Establishing that for n sufficiently large $A_n = 0$.

Part 1. Analytic Part of Proof–An upper bound for $|A_n|$

The analytic estimate for $|A_n|$ follows from the representation

$$A_n = \frac{1}{2\pi i}\int_{\gamma_n} \frac{e^{\pi\zeta}}{\zeta(\zeta - z_1)\dots(\zeta - z_n)}d\zeta.$$

Therefore:

$$\begin{aligned}
|A_n| &= |\frac{1}{2\pi i}\int_{\gamma_n} \frac{e^{\pi\zeta}}{\zeta(\zeta - z_1)\dots(\zeta - z_n)}d\zeta| \\
&\leq \frac{1}{2\pi} \times (\text{length of the contour } \gamma_n) \times \max_{\zeta\in\gamma_n} \frac{|e^{\pi\zeta}|}{|\zeta||\zeta - z_1|\dots|\zeta - z_n|} \\
&\leq \frac{1}{2\pi} \times (\text{length of the contour } \gamma_n) \times \frac{\max_{\zeta\in\gamma_n}|e^{\pi\zeta}|}{\min_{\gamma\in\gamma_n}|\zeta||\zeta - z_1|\dots|\zeta - z_n|}.
\end{aligned}$$

To provide a reasonably small upper bound for $|A_n|$ Gelfond needed to understand the possible contours γ_n that would encircle the Gaussian integers appearing in the denominator of the integral representation of A_n. In order to obtain a small upper bound for $|A_n|$ Gelfond needed the length of the contour to be as small as possible, but he also needed each of

$$\max_{\zeta\in\gamma_n}\{|e^{\pi\zeta}|\} \quad \text{and} \quad \frac{1}{\min_{\zeta\in\gamma_n}|\zeta||\zeta - z_1|\dots|\zeta - z_n|}. \tag{3}$$

to be small. Clearly any estimate for either of these quantities will depend on the choice of the contour of integration.

Since the absolute values of the ordered Gaussian integers is nondecreasing, before specifying γ_n Gelfond needed to estimate $|z_n|$, and so know how large of a contour to use, Gelfond referred to some estimates due to E. Landau. If we temporarily let $G(r)$ denote the number of Gaussian integers $x_k + y_k i$ with $x_k^2 + y_k^2 \leq r^2$ then it is not too difficult to derive the estimate Gelfond used (one simply shows that $G(r)$ is greater than the area of an appropriately chosen smaller circle and less than the area of an appropriately chosen larger circle (see exercises)). The result is that for $r > \sqrt{2}$,

$$\pi(r-\sqrt{2})^r \leq G(r) \leq \pi(r+\sqrt{2})^2.$$

From this it follows, see exercises, that the nth Gaussian integer, in Gelfond's ordering, satisfies:

$$|z_n| = \sqrt{\frac{n}{\pi}} + o(\sqrt{n}), \text{ where } \lim_{n\to\infty} \frac{o(\sqrt{n})}{\sqrt{n}} = 0.$$

The above estimate told Gelfond that he could take the contour of integration to be the circle of some radius greater than a constant times $\sqrt{\frac{n}{\pi}}$, Gelfond used the relatively large radius of n. With this contour it is simple to estimate the first expression in (3):

$$\max_{\zeta\in\gamma_n}\{|e^{\pi\zeta}|\} \leq e^{\pi \max\{Re(\zeta):\zeta\in\gamma_n\}} = e^{\pi n} \tag{4}$$

To estimate the second expression in (3) we need an estimate for the minimum distance from each of the first n Gaussian integers, $z_1, z_2, \ldots, z_n$ and the points of the circle γ_n, and we want this minimum to not be too small. A need for such an estimate points to one reason Gelfond took the contour of integration to have a larger radius than would be needed to simple contain the first n Gaussian integers. From the estimate for $|z_n|$, above, we see that for n sufficiently large:

$$|z_n| \leq \sqrt{\pi}\sqrt{\frac{n}{\pi}} = \sqrt{n},$$

so for any $1 \leq i \leq n, \min\{|\zeta - z_i| : \zeta \in \gamma_n\} \geq n - \sqrt{n} \geq \frac{1}{2}n$, for n sufficiently large. Therefore we have:

$$\max_{\zeta\in\gamma_n} \frac{1}{|\zeta||\zeta-z_1|\ldots|\zeta-z_n|} = \frac{1}{\min_{\zeta\in\gamma_n}|\zeta||\zeta-z_1|\ldots|\zeta-z_n|} \leq \big(\frac{2}{n}\big)^{n+1}$$

Putting all of the above estimates together we obtain, for n sufficiently large,

$$\begin{aligned}|A_n| &= |\frac{1}{2\pi i}\int_{\gamma_n} \frac{e^{\pi\zeta}}{\zeta(\zeta-z_1)\ldots(\zeta-z_n)}d\zeta| \\ &\leq \frac{1}{2\pi} \times (\text{length of the contour } \gamma_n) \times \max_{\zeta\in\gamma_n} \frac{|e^{\pi\zeta}|}{|\zeta||\zeta-z_1|\ldots|\zeta-z_n|} \\ &\leq \frac{1}{2\pi} \times 2\pi n \times e^{\pi n} \times \big(\frac{2}{n}\big)^{n+1} \leq e^{\log n + n - (n+1)\log(n/2)}.\end{aligned}$$

Warning: If we were to further simplify this estimate to something like $e^{-1/2n\log n}$, for n sufficiently large, Gelfond's proof will fail.

Part 2. Algebraic Part of Proof–A lower bound for $|A_n|$ for those n for which $A_n \neq 0$.

We begin with a simple application of the Residue Theorem that allows us to express A_n as an algebraic number:

$$\begin{aligned} A_n &= \frac{1}{2\pi i}\int_{\gamma_n} \frac{e^{\pi\zeta}}{\zeta(\zeta-z_1)\ldots(\zeta-z_n)}d\zeta \\ &= \sum_{k=0}^{n}\left\{ \text{residue of } \frac{e^{\pi z}}{z(z-z_1)\ldots(z-z_n)} \text{ at } z=z_k\right\} \\ &= \sum_{k=0}^{n} \frac{e^{\pi z_k}}{\prod_{j=0,j\neq k}^{n}(z_k-z_j)}. \end{aligned}$$

If for each of the ordered Gaussian integers we use the notation $z_k = x_k + y_k i$, where x_k and y_k are ordinary integers which may be positive, negative, or zero, then

$$\begin{aligned} A_n &= \sum_{k=0}^{n} \frac{e^{\pi z_k}}{\prod_{\substack{j=0\\ j\neq k}}^{n}(z_k-z_j)} \\ &= \sum_{k=0}^{n} \frac{e^{\pi(x_k+y_k i)}}{\prod_{\substack{j=0\\ j\neq k}}^{n}(z_k-z_j)} \\ &= \sum_{k=0}^{n} \frac{(e^{\pi})^{x_k}(-1)^{y_k}}{\prod_{\substack{j=0\\ j\neq k}}^{n}(z_k-z_j)}. \end{aligned}$$

This equation shows that for each n, A_n is an algebraic number because each of the summands

$$\frac{(e^{\pi})^{x_k}(-1)^{y_k}}{\prod_{\substack{j=0\\ j\neq k}}^{n}(z_k-z_j)} \tag{5}$$

is a ratio of algebraic numbers.

Of course the (algebraic) norm of a **nonzero** algebraic *integer* is an ordinary integer that is not equal to zero, but the algebraic norm of an algebraic number that is not an algebraic integer is simply a rational number. In order to obtain an integer from A_n we need to first multiply through by its denominator. The denominator of each of the summands for the above representation of A_n is a product of differences of Gaussian integers. Since $\mathbf{Z}[i]$ is a ring, these denominators are themselves Gaussian integers. We need to better understand both the denominators and numerators in order to find an appropriate integers to multiply A_n by in order to obtain an algebraic integer. It is easier to see what is going on if we simplify our notation. Following Gelfond put

$$\prod_{\substack{j=0\\ j\neq k}}^{n}(z_k-z_j) = \omega_{n,j}.$$

Then

$$A_n = \frac{(e^\pi)^{x_0}(-1)^{y_0}}{\omega_{n,0}} + \frac{(e^\pi)^{x_1}(-1)^{y_1}}{\omega_{n,1}} + \cdots + \frac{(e^\pi)^{x_n}(-1)^{y_n}}{\omega_{n,n}}. \tag{6}$$

The natural thing to try is to let Ω_n equal the product of all of the denominators in(6), so that $\Omega_n \times A_n$ is an algebraic integer. From the expression

$$\Omega_n = \prod_{j=0}^{n} \omega_{n,j} = \prod_{k=0}^{n} \prod_{\substack{j=0 \\ j \neq k}}^{n} (z_k - z_j),$$

it is not too difficult to estimate $|\Omega_n|$. Since it is a product of $n(n-1)$ positive integers each of which is less than $2|z_n|$, and $|z_n| = \sqrt{n/\pi} + o(\sqrt{n})$, we see that for n sufficiently large,

$$|\Omega_n| \leq \left(\sqrt{n}\right)^{n(n-1)} = e^{\frac{1}{2}n(n-1)\log n}.$$

We will see below that this estimate is not nearly small enough for Gelfond to conclude that the upper bound and lower bound for $|A_n|$ contradict each other. Gelfond may have tried this but then he settled on a more subtle choice for Ω_n that ultimately does work.

The ring $\mathbf{Z}[i]$ is a unique factorization domain, so the notion of the least common multiple of a collection of Gaussian integers makes sense. In another paper, also published in 1929, Gelfond studied the distribution of the irreducible elements in $\mathbf{Z}[i]$ and concluded that $|\Omega_n|$ is not as large as the simplistic estimate above. Gelfond established the quite precise estimate for

$$\Omega_n = L.C.M.\Big\{(z_1 - z_2)(z_1 - z_3)\ldots(z_1 - z_n), \\ (z_2 - z_1)(z_2 - z_3)\ldots(z_2 - z_n), \ldots, (z_n - z_1)(z_n - z_2)\ldots(z_n - z_{n-1})\Big\}.$$

of

$$|\Omega_n| \leq e^{\frac{1}{2}n\log n + 163n + O(\sqrt{n})}.$$

If we recall that $\prod_{j=0, j\neq k}^{n}(z_k - z_j) = \omega_{n,j}$ we have a precise representation for $\Omega_n \times A_n$:

$$\Omega_n \times A_n = \frac{\Omega_n}{\omega_{n,0}}(e^\pi)^{x_0}(-1)^{y_0} + \frac{\Omega_n}{\omega_{n,1}}(e^\pi)^{x_1}(-1)^{y_1} + \cdots + \frac{\Omega_n}{\omega_{n,n}}(e^\pi)^{x_n}(-1)^{y_n}.$$

It is worth making two observations about the above expression:

Observation 1. Each algebraic number $\dfrac{\Omega_n}{\omega_{n,k}}$ is an element of $\mathbf{Z}[i]$, so is an algebraic integer.

Observation 2. In Gelfond's ordering for the elements $z_k = x_k + y_k i$, in $\mathbf{Z}[i]$, x_k may be positive, negative, or zero. Thus each of the numerators in the above expression involves can involve either e^{π} or $e^{-\pi}$.

These two observations tell us that $\Omega_n \times A_n$ is an integral polynomial expression in $e^{\pi}, e^{-\pi}$ and i. It is possible to simplify things a bit by multiplying through by an appropriately high power of e^{π}. Specifically, if we let $r_n = \max_{0 \leq k \leq n}\{|x_k|\}$, and note for later use that $r_n \leq \sqrt{n}$, then $(e^{\pi})^{r_n}\Omega_n A_n$ is an integral polynomial expression in e^{π} and i :

$$(e^{\pi})^{r_n}\Omega_n A_n = \frac{\Omega_n}{\omega_{n,0}}(e^{\pi})^{r_n+x_0}(-1)^{y_0} + \frac{\Omega_n}{\omega_{n,1}}(e^{\pi})^{r_n+x_1}(-1)^{y_1} + \cdots + \frac{\Omega_n}{\omega_{n,n}}(e^{\pi})^{r_n+x_n}(-1)^{y_n}.$$

Finally, if we let δ denote a denominator for e^{π}, we then have the algebraic integer:

$$\begin{aligned} P_n(i, e^{\pi}) &= (\delta)^{2\sqrt{n}}(e^{\pi})^{r_n}\Omega_n A_n \\ &= (\delta)^{2\sqrt{n}-(r_n+x_0)}\frac{\Omega_n}{\omega_{n,0}}(\delta e^{\pi})^{r_n+x_0}(-1)^{y_0} + (\delta)^{2\sqrt{n}-(r_n+x_1)}\frac{\Omega_n}{\omega_{n,1}}(\delta e^{\pi})^{r_n+x_1}(-1)^{y_1} \\ &\quad + \cdots + (\delta)^{2\sqrt{n}-(r_n+x_n)}\frac{\Omega_n}{\omega_{n,n}}(\delta e^{\pi})^{r_n+x_n}(-1)^{y_n}, \end{aligned}$$

where $P_n(x, y)$ is the obvious integral polynomial. Our goal is to calculate the algebraic norm of the algebraic integer $P_n(i, e^{\pi})$, which we will denote by $N(P_n(i, e^{\pi}))$.

Since $P_n(i, e^{\pi}) \neq 0$ we know that $Norm(P_n(i, e^{\pi}))$ is a nonzero integer. In order to get a handle on $Norm(P_n(i, e^{\pi}))$ we denote the conjugates of e^{π} by $\theta_1(= e^{\pi}), \theta_2, \ldots, \theta_d$. Then an integral power of $Norm(P_n(i, e^{\pi}))$ is given by the product:

$$N = P_n(i, \theta_1)P_n(-i, \theta_1)\Big(\prod_{j=2}^{d} P_n(i, \theta_j)\Big)\Big(\prod_{j=2}^{d} P_n(-i, \theta_j)\Big). \qquad (7)$$

If $A_n \neq 0$ then $N \neq 0$.

We will use our earlier analytic work to provide an upper bound for the first factor, $|P_n(i, \theta_1)|$, and algebraic information about the Gaussian integers to estimate the absolute values of each of the other factors.

Our earlier analytic estimate for $|A_n|$, combined with Gelfond's estimate for $|\Omega_n|$ and the estimates $r_n \leq \sqrt{n}$ and $|x_n| \leq \sqrt{n}$ yields:

$$|P(i, \theta_1)| \leq e^{-\frac{1}{2}n\log n + 170n}, \text{ provided } n \text{ is sufficiently large.} \qquad (8)$$

Each of the other $2d-1$ factors in (7) is estimated by the triangle inequality. The most difficult terms to estimate are the ratios $\dfrac{\Omega_n}{\omega_{n,k}}$. We have already seen that in a different paper Gelfond provided an estimate for the numerator $|\Omega_n|$. To provide a lower bound on the denominator, Gelfond uses an estimate from another mathematician, Seigo Fukasawa, who, in 1926, showed that for n sufficiently large,

$$|\omega_{n,k}| > e^{\frac{1}{2}n\log n - 10n}.$$

Therefore, provided n is sufficiently large, we have

$$|\frac{\Omega_n}{\omega_{n,k}}| \leq e^{\frac{1}{2}n\log n + 164n - (\frac{1}{2}n\log n - 10n)} \leq e^{174n}.$$

Putting all of these estimates together we see that each of the other factors in (7) we have:

$$|P_n(\pm i, \theta_j)| \leq (n+1)e^{174n}(\delta e^{\pi})^{2\sqrt{n}} \leq e^{175n}, \text{ for } n \text{ sufficiently large.}$$

Finally, we have for the nonzero integer N,

$$0 < |N| \leq e^{-\frac{1}{2}n\log n + (2d-1)175n}. \tag{9}$$

The only way to avoid the contradiction presented by the above inequalities is to conclude that our assumption that $A_n \neq 0$ must be wrong. This leads us to the conclusion that for n sufficiently large, say $n > N^*, A_n = 0$. This tells us that for all $n > N^*$ we have the representation for the function $e^{\pi z}$:

$$e^{\pi z} = A_0P_0(z) + A_1P_1(z) + \cdots + A_{N^*}P_{N^*}(z) + R_n(z).$$

We take the contour of integration in the integral representation for $R_n(z)$ to be a circle with radius n. Since for any $z, R_n(z) \to 0$ as $n \to \infty$ it follows that $e^{\pi z}$ equals the polynomial $A_0P_0(z) + A_1P_1(z) + \cdots + A_{N^*}P_{N^*}(z)$ and so is not a transcendental function. It follows that e^z is not a transcendental function, which is our long sought contradiction.

Two other partial solutions

The title of this chapter is "Three partial solutions to Hilbert's seventh problem", of which Gelfond's was the first. One year later R.O. Kuzmin gave his partial solution to Hilbert's problem. His paper began:

In December's installment of the journal Comptes Rendus de l'Academie des Sciences de Paris there was an interesting article by A. Gelfond, in which the author obtained a new result in the theory of transcendental numbers with the help of extremely clever reasoning. ... The method, which I use here, is closely based on the method of A.O. Gelfond (which I do not know very well, as he published only in Japanese journals, which are inaccessible to me). Perhaps for this reason my methodology is simpler and more elementary. In particular I proceed without complex functional analysis.

Theorem (Kuzmin 1930) $2^{\sqrt{2}}$ *is transcendental.*

We mentioned that Gelfond's approach could be used to establish somewhat more, Kuzmin's actually established the more general result that his method could establish: For any positive rational number r that is not a perfect square and for any algebraic number $\alpha \neq 0, 1$, $\alpha^{\sqrt{r}}$ is transcendental. For simplicity we only look at his proof of the transcendence of $2^{\sqrt{2}}$.

Both the gross structure, and the nature of the details, in Kuzmin's proof are strikingly similar to Gelfond's proof. In broad outline Kuzmin starts with the assumption that $2^{\sqrt{2}}$ is algebraic and studies this value by considering the real-valued function $2^z = e^{\log(2)x}$. He then approximates 2^z using the Lagrange interpolation formula (not using the Gaussian integers but the numbers $\{a + b\sqrt{2} : a, b$ integers, not both zero$\}$, which he orders as $z_1, z_2, \ldots$. Then, using the notation $P_k(z) = (z - z_1)(z - z_2) \ldots (z - z_k)$, Kuzmin had that for each n,

$$2^z = \sum_{k=1}^{n} \frac{P_n(z)}{z - z_k} \frac{2^{z_k}}{P_n'(z_k)} + \frac{P_n(z)}{n!} 2^{\zeta} (\log 2)^n,$$

where ζ lies between the smallest and the largest of z_1 through z_n.

But $P_n'(z) = \sum_{r=1}^{n} \prod_{\ell \neq r} (z - z_\ell)$ which implies that $P_n'(z_k) = \prod_{\ell \neq k} (z_k - z_\ell)$. Therefore, for $z_0 \notin \{z_1, \ldots, z_n\}$

$$2^{z_0} = \sum_{k=1}^{n} \frac{P_n(z_0) 2^{z_k}}{(z_0 - z_k) \prod_{\ell \neq k} (z_k - z_\ell)} + \frac{P_n(z_0)}{n!} 2^{\zeta} (\log 2)^n.$$

Dividing through by $P_n(z_0)$, and rewriting:

$$\sum_{k=0}^{n} \frac{2^{z_k}}{\prod_{\ell \neq k} (z_k - z_\ell)} = \frac{2^{\zeta} (\log 2)^n}{n!}.$$

1. Multiplying the left-hand side of this equality by the least common multiple of the denominators and then multiplying by an algebraic denominator yields a nonzero algebraic integer.

2. Taking n sufficiently large the right hand side quantity has a small absolute value.
3. Taking the algebraic norm of the left-hand side produces a nonzero integer whose absolute value is less than 1.

These two estimates contradict, so the error term in Lagrange Interpolation equals 0. It follows that the transcendental function $f(z) = 2^z$ is a polynomial function. This contradiction establishes the result.

A couple of years later a German mathematician, Karl Boehle, published a paper that generalized the results of both Gelfond and Kuzmin, yet still fell far short of *solving* Hilbert's seventh problem. In his paper Boehle acknowledged that his work built on Gelfond and Kuzmin's:

In 1929 A. Gelfond demonstrated the transcendence of the number α^β when α is an irrational, neither 0 nor 1, algebraic number and β is a quadratic irrationality. C.L.Siegel showed in a Number Theory Seminar in February 1930 that α^β is transcendental when β is a real quadratic irrationality. R.A. Kuzmin proved this also (Bulletin de l'Academie des Sciences URS, Leningrad 1930, No. 6).

Theorem (Boehle 1932) *Suppose $\alpha \neq 0, 1$ and β are algebraic numbers, $d = deg(\beta) \geq 2$. Then at least one of the numbers*

$$\alpha^\beta, \ldots, \alpha^{\beta^{d-1}} \text{ is transcendental.}$$

Proof (Main Idea). Assume all of the numbers $\alpha, \alpha^\beta, \ldots, \alpha^{\beta^{d-1}}$ are algebraic. Boehle examined the function $f(z) = \alpha^z$ at points $n_1 + n_2\beta + \cdots + n_d\beta^{d-1}$

Note that for $d = 2$, so for example if $\beta = \sqrt{n}$ is irrational then Boehle re-established the transcendence of $\alpha^{\sqrt{n}}$.

Gelfond's theorem follows from Boehle's upon taking $\beta = \sqrt{-r}$, where r is a positive, rational numbers. Boehle's theorem then implies that one of the two numbers

$$\alpha^{\sqrt{-r}} \text{ or } \alpha^{(\sqrt{-r})^2} = \alpha^{-r},$$

must be transcendental. Since the second of these numbers is algebraic the first of them, the one Geldfond addressed, must be the transcendental one.

The deduction of Kuzmin's result from Boehle's is the similar, with $\sqrt{r}$ replacing $\sqrt{-r}$.

Exercises

1. Show that e^z is a transcendental function.

2. Let a and b be complex numbers. Show that the functions e^{az} and e^{bz} are algebraically independent if and only if a/b is irrational. (We will use this result in the next chapter.)

3. Convince yourself that the estimate for $G(r)$ given above is correct. (Hint: Let $n = G(r)$. Let z_k be one of the first n Gaussian integers and associate with z_k the unit square whose vertices are Gaussian integers and whose lower left corner is the Gaussian integer z_k. We want to take a smaller circle, of radius $r' < r$, so that its area is less than n. It suffices to let $r' = r - \sqrt{2}$. Similarly we want to take a larger radius, $r'' > r$, so than the radius of this larger circle exceeds n. This time it suffices to take $r'' = r + \sqrt{2}$. It follows that

$$\pi(r - \sqrt{2})^2 < G(r) < \pi(r + \sqrt{2})^2.)$$

4. Conclude from the estimate in problem 3 (above) that if z_n denotes the nth Gaussian integer in Gelfond's ordering then

$$|z_n| = \sqrt{\frac{n}{\pi}} + o(\sqrt{n}).$$

(Hint: If $G(r) = n$ let $r = |z_n|$. Then we have $n = \pi|z_n|^2 + O(|z_n|)$. Therefore

$$n = \pi|z_n|^2 + O(|z_n|),$$

which leads to

$$\Big|\frac{n}{\pi} - |z_n|^2\Big| = \Big|\sqrt{\frac{n}{\pi}} - |z_n|\Big|\Big|\sqrt{\frac{n}{\pi}} + |z_n|\Big| = O(|z_n|).)$$

5. Verify the equality in the displayed line (4).

6. Derive the naive estimate for $|\Omega_n|$.

7. Explain why we wrote the N is an *integral power* of the norm of $P_n(i.e^{\pi})$.

8. Convince yourself the N, on page 8, is a nonzero integer.

9. Give an restructured outline of Gelfond's proof so that the proof concludes with a contradiction

$$0 < \text{ rational integer } < 1.$$